Closing Thu: 13.3(part 1),
Closing Tue: 13.3(part 2), 13.4
Closing Next Thu: 14.1, 14.3 (part 1)
Midterm 1 will be returned Tuesday.
And updated grades will be posted by the end of next week.

13.3 (part 2) TNB Frame

Today we define our last few 3D measurement tools.

First, the **normal plane** to $\vec{\boldsymbol{r}}(t)$ at a point is the plane that goes through the point and is orthogonal to the curve.

Example: Find the normal plane at $t = \pi$ for

$$\vec{\boldsymbol{r}}(t) = < 2\sin(3t), t, 2\cos(3t) >$$

(See visual from ebook)

Finding normal vectors (TNB-Frame):

As we proved the other day in class, $\vec{\boldsymbol{T}}'(t)$ is always orthogonal to $\vec{\boldsymbol{T}}(t)$.

Not only is it orthogonal, it also points `inwardly' relative to whichever way you are curving.

If we make this inward pointing vector a unit vector, then we call it:

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$

We also define

$$\vec{\boldsymbol{B}}(t) = \vec{\boldsymbol{T}}(t) \times \vec{\boldsymbol{N}}(t) = \text{binormal}$$

Note that $\vec{\boldsymbol{B}}(t)$ already has length one (why?).

Some TNB Facts:

- All have length one.

- **Normal plane** is parallel to $\vec{\boldsymbol{N}}(t)$ and $\vec{\boldsymbol{B}}(t)$ And is orthogonal to $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{r}}'(t)$.

- $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ point in the tangent and inward directions, respectively. They give a good approximation of the "plane of motion".

- This "plane of motion" that goes through a point on the curve and is parallel to $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ is called the "**osculating plane**" ("osculating" means "kissing")

- $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$ are ALL parallel to the osculating plane.

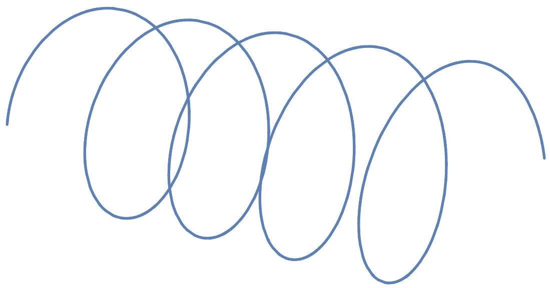
- $\vec{\boldsymbol{B}}(t)$ is orthogonal to the osculating plane, it is also orthogonal to ALL the vectors $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$

Example:

$$\vec{\boldsymbol{r}}(t) = < 2\sin(3t)\,, t, 2\cos(3t) >$$

Find

1. $\vec{\boldsymbol{T}}(\pi)$
2. $\vec{\boldsymbol{N}}(\pi)$
3. $\vec{\boldsymbol{B}}(\pi)$
4. Find the osculating plane at t = π



A `sneaky' old exam question:
Consider the curve

$$\vec{\boldsymbol{r}}(t) = < (t^2 - 2)^2, t^4, t^2 >$$

a) Compute $\vec{\boldsymbol{T}}(t)$ for general t.

b) Show that the curve lies in the plane

x−y+4z = 4.

c) Find one (non-zero) vector that is parallel to $\vec{\boldsymbol{B}}(1)$.

Hint: Think about what b) means for the osculating plane and for the position of the vectors $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, $\vec{\boldsymbol{B}}(t)$. You can use those insights to solve c) and d) with very little calculations.

Summary of 3D Curve Measurement Tools:

Given $\vec{\boldsymbol{r}}(t) = < x(t), y(t), z(t) >$

$\vec{\boldsymbol{r}}'(t) =$ a tangent vector

$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(t)| dt =$ distance (arc length)

$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3} =$ curvature

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$

$$\vec{\boldsymbol{B}}(t) = \vec{\boldsymbol{T}}(t) \times \vec{\boldsymbol{N}}(t) = \text{binormal}$$

Tangent Line:
Through curve in direction of tangent.
Normal Plane:
Through curve orthogonal to tangent.
Osculating Plane:
Through curve parallel to $\vec{\boldsymbol{r}}'(t)$ and $\vec{\boldsymbol{r}}''(t)$